Fundamental Theorem of Calculus (Continued)

① Suppose $f(t)$ be a continuous function over $[a,b]$, then

$$F(x) = \int_a^x f(t)\,dt$$

Then $F(x)$ is an antiderivative of f.

② Suppose, $f(t)$ is continuous function on $[a,b]$ & F is an antiderivative of f. Then

$$\int_a^b f(t)\,dt = F(t)\Big|_a^b = F(b) - F(a)$$

Note:- a, b are constants.

General Version:-

Suppose, f is continuous & g and h be differentiable.

Then,
$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = F'(h(x))\cdot h'(x) - F'(g(x))\cdot g'(x)$$

where F is an antiderivative of f.

Example:-

① $\int_1^3 f(x)\,dx = 7$, $\int_1^6 f(x)\,dx = 17$ & $\int_1^3 g(x)\,dx = -3$, $\int_1^6 g(x)\,dx = 0$

Q. Find $\int_1^3 [5f(x) - 3g(x)]\,dx$

$\rightsquigarrow \int_1^3 5f(x)\,dx - \int_1^3 3g(x)\,dx = 5\int_1^3 f(x)\,dx - 3\int_1^3 g(x)\,dx = 5(7) - 3(-3)$

$= 35 + 9 = 44$

Q. Find $\int_3^6 [2f(x) + g(x)]\,dx$

$\rightsquigarrow$ Note:- $\int_1^3 f(x)\,dx + \int_3^6 f(x)\,dx = \int_1^6 f(x)\,dx$

$\Rightarrow 7 + \int_3^6 f(x)\,dx = 17 \Rightarrow \int_3^6 f(x)\,dx = 10$

Similarly, $\int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx = \int_1^6 g(x)\,dx$

$\Rightarrow -3 + \int_3^6 g(x)\,dx = 0 \Rightarrow \int_3^6 g(x)\,dx = 3$

Then, $\int_3^6 [2f(x) + g(x)]\,dx = 2\int_3^6 f(x)\,dx + \int_3^6 g(x)\,dx = 2(10) + (3)$
$= 23$

② Calculate : $\int_0^\pi 2\sin x\,dx$ & $\int_{\pi/2}^{\pi} 2\sin x\,dx$

Here, $f(x) = 2\sin x$. Antiderivative of $f(x)$ is $-2\cos x$.

Then $\int_0^\pi 2\sin x\,dx = -2\cos x \Big|_0^\pi = (-2\cos\pi) - (-2\cos 0)$
$= 2 - (-2) = 4$

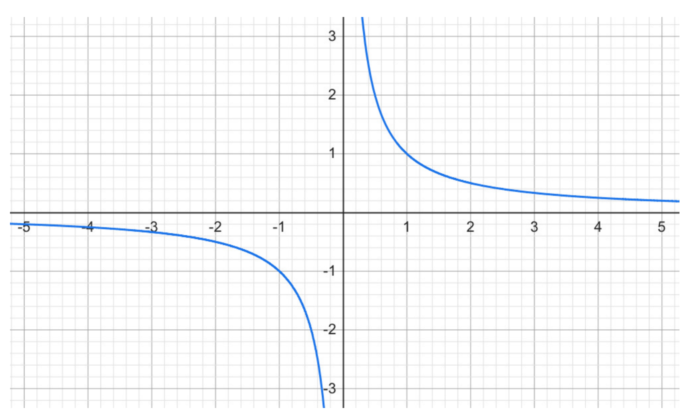
$\int_{\pi/2}^{\pi} 2\sin x\,dx = -2\cos x \Big|_{\pi/2}^{\pi} = (-2\cos\pi) - (-2\cos\pi/2)$
$= 2 - (0) = 2$

③ Find $\int_{-1}^{1} \frac{1}{x}\,dx$

Here, $f(x) = \frac{1}{x}$

Note:- at $x = 0$, $f(x)$ is not continuous.

Hence, FTC is not applicable.



④ Find $\int_1^3 \frac{t^3+4t^2+4}{t}\,dt$

$\rightsquigarrow$ $\int_1^3 \left(\frac{t^3}{t}+\frac{4t^2}{t}+\frac{4}{t}\right)dt = \int_1^3 \left(t^2+4t+\frac{4}{t}\right)dt.$

Now, $f(t) = t^2+4t+\frac{4}{t}$

So antiderivative $F(t) = \frac{t^3}{3} + 4\cdot\frac{t^2}{2} + 4\ln|t|$

$= \frac{t^3}{3} + 2t^2 + 4\ln|t|$

Then, $\int_1^3 f(t)\,dt = F(3) - F(1)$

$= \left(\frac{3^3}{3} + 2\cdot 3^2 + 4\ln|3|\right) - \left(\frac{1^3}{3} + 2\cdot 1^2 + 4\ln|1|\right)$

$= (9+18+4\ln 3) - \left(\frac{1}{3}+2+0\right)$

$= 25 - \frac{1}{3} + 4\ln 3$

$= \frac{74}{3} + 4\ln 3$

⑤ Evaluate $F'(4)$, where $F(x) = \int_4^x \sqrt{t^3}\,dt$

$\rightsquigarrow$ By FTC, $F'(x) = \sqrt{x^3}$

Then $F'(4) = \sqrt{4^3} = 4\sqrt{4} = 4(2) = 8.$

⑥ Calculate the derivative of $\Psi(x) = \int_{1.5}^x \sqrt{(t^2+3t)}\,dt$, at $x=3$.

(psi)

⟿ Here, $\psi(x) = \int_{1.5}^{x} \sqrt{t^2+3t}\, dt$.

Then by FTC, $\psi'(x) = \sqrt{x^2+3x}$

Now, $\psi'(3) = \sqrt{3^2+3(3)} = \sqrt{9+9} = \sqrt{9(2)}$

$= \sqrt{9} \cdot \sqrt{2}$

$= 3\sqrt{2}$

⑦ Find the derivative of $F(x) = \int_{0}^{x^2} \sqrt{(1+t^3)}\, dt$

⟿ Note: To apply FTC, we need x inplace of x^2

$$\int_{0}^{x^2} \sqrt{(1+t^3)}\, dt$$

We can tackle that by simply considering a new variable $u = x^2$, then

$$\boxed{\int_{0}^{u} \sqrt{(1+t^3)}\, dt}$$

But now it became a new function.

So let, $G(u) = \int_{0}^{u} \sqrt{(1+t^3)}\, dt$

Observation,
$G(x^2) = F(x)$

Now we apply FTC & get,

$$G'(u) = \sqrt{(1+u^3)}$$

Then, $F'(x) = \frac{d}{dx}\left(G(x^2)\right)$

$$= G'(x^2) \cdot 2x$$

$$= \sqrt{1+(x^2)^3} \cdot 2x$$

$$= 2x\sqrt{1+x^6}$$